Assignment 5: Differential Equations: Solutions

1. (a) The auxiliary equation is $x^2 - 2x + 4 = 0$. Using the quadratic formula, one can find that the roots are $x = 1 \pm i\sqrt{3}$. Thus the general solution is

$$y = e^x(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

   (b) The complementary equation has auxiliary equation $x^2+5x-3 = 0$, which factors as $x = \frac{1}{2}, x = -3$. Thus the complementary equation has solution

$$y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-3x}$$

   To find a particular solution, since x^2+1 is a polynomial of degree 2, we use $y_p = Ax^2 + Bx + C$. This has derivative $y_p' = 2Ax + B$ and second derivative $y_p'' = 2A$. Putting this into the original DE gives

$$4A + 10Ax + 5B - 3Ax^2 - 3Bx - 3C = x^2 + 1$$

   Comparing coefficients gives the equations

$$3A = 1,\ 10A - 3B = 0,\ 4A + 5B - 3C = 1$$

   Solving these gives $A = -\frac{1}{3}$, $B = -\frac{10}{9}$, $C = -\frac{71}{27}$. Thus the general solution is

$$y = y_c + y_p = c_1 e^{\frac{1}{2}x} + c_2 e^{-3x} + \left(-\frac{1}{3}x^2 + -\frac{10}{9}x + -\frac{71}{27}\right).$$

   (c) The complementary equation has auxiliary equation $x^2+x-2 = 0$, which has roots $x = 1, x = -2$. Thus the complementary equation has solution

$$y_c = c_1 e^{-2x} + c_2 e^x$$

   We will need to two particular solutions: y_{p1} for e^x and y_{p2} for $\sin(2x)$.

For y_p1, we have to use $y_p1 = Axe^x$ since Ae^x is already a solution to the complementary equation. y_p1 has derivative $y_{p1} = Ae^x + Axe^x$ and second derivative $y''_{p1} = 2Ae^x + Axe^x$. Putting this into the DE gives

$$2Ae^x + Axe^x + Ae^x + Axe^x - 2Axe^x = e^x$$

comparing coefficients gives $3A = 1$, so $A = \frac{1}{3}$. So $y_{p1} = \frac{xe^x}{3}$.

For $\sin(2x)$, we try $y_{p2} = A\sin(2x)+B\cos(2x)$. This has derivative

$$y'_{p2} = 2A\cos(2x) - 2B\cos(2x)$$

and second derivative

$$y''_{p2} = -4a\sin(2x) - 4B\cos(2x)$$

Putting this into the DE gives

$$(-6A - 2B)\sin(2x) + (-6B + 2A)\cos(2x) = \sin(2x)$$

This gives equation $-6A-2B = 1$ and $-6B+2A = 0$. Solving this gives $B = -\frac{1}{20}$ and $A = -\frac{3}{20}$. Thus $y_{p2} = \frac{3}{20}\cos(2x) - \frac{1}{20}\sin(2x)$. Thus the DE has solution

$$y = y_c + y_{p1} + y_{p2} = c_1e^{-2x} + c_2e^x + \frac{xe^x}{3} - \frac{3}{20}\cos(2x) - \frac{1}{20}\sin(2x)$$

2. The complementary equation has auxiliary equation $x^2 - x + \frac{1}{4} = 0$, which factors as $(x - \frac{1}{2})^2 = 0$. Thus the complementary equation has solution

$$y_c = c_1e^{\frac{1}{2}x} + c_2xe^{\frac{1}{2}x}$$

Since the polynomial x is of degree, we look for a particular solution $y_p = Ax + B$. This has derivative $y'_p = A$ and second derivative $y''_p = 0$. Putting this into the DE gives

$$-4A + Ax + B = x$$

This gives equations

$$-4A + B = 0 \text{ and } A = 1$$

Solving for B gives $B = 4$. Thus the general solution is

$$y = y_c + y_p = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x} + x + 4$$

Using $y(0) = 3$ gives $c_1 + 4 = 3$, so $c_1 = -1$. We calculate the derivative of y as

$$y' = \frac{1}{2} c_1 e^{\frac{1}{2}x} + c_2 e^{\frac{1}{2}x} + \frac{1}{2} c_2 x e^{\frac{1}{2}x} + 1$$

So $y'(0) = 1$ gives $\frac{1}{2}c_1 + c_2 + 1 = 1$, so $c_2 = \frac{1}{2}$. Thus the solution is

$$y = -e^{\frac{1}{2}x} + \frac{1}{2} x e^{\frac{1}{2}x} + x + 4$$

3. The complementary equation has auxiliary equation $x^2 - 5x + 6 = 0$, which has roots $x = 2$, $x = 3$. Thus the complementary equation has solution

$$y_c = c_1 e^{2x} + c_2 e^{3x}$$

Using the equation for u' gives

$$u' = \frac{-g y_2}{a(y_1 y_2' - y_2 y_1')} = \frac{-e^{3x}(e^{3x})}{e^{5x}} = -e^x$$

Integrating gives $u = -e^x$.

Using the equation for z' gives

$$z' = \frac{g y_1}{a(y_1 y_2' - y_2 y_1')} = \frac{e^{3x}(e^{2x})}{e^{5x}} = 1$$

Integrating gives $z = 1$. Thus the particular solution is

$$y_p = u y_1 + z y_2 = -e^x(e^{2x}) + x e^{3x} = (x-1)e^{3x}$$

Thus the general solution is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{3x} + (x-1)e^{3x}$$

4. We attempt to find a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

We can calculate the first and second derivatives of this power series as

$$y' = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n \text{ and } y'' = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

We then substitute these into the DE to get

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - x\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - 2\sum_{n=0}^{\infty}c_nx^n = 0$$

or

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty}-(n+1)c_{n+1}x^{n+1} + \sum_{n=0}^{\infty}-2c_nx^n = 0$$

Then comparing the x^0 (constant) coefficient on both sides, we get

$$2c_2 - 2c_0 = 0$$

so $c_2 = c_0$. Comparing the x^1 coefficient gives

$$6c_3 - c_1 - 2c_1 = 0$$

or $c_3 = \frac{1}{2}c_1$. Continuing on in this fashion, looking at the coefficients of higher powers of x, we get the relations

$$c_4 = \frac{1}{3}c_0,\ c_5 = \frac{1}{2\cdot 4}c_1,\ c_6 = \frac{1}{3\cdot 5}c_0,\ c_7 = \frac{1}{2\cdot 4\cdot 6}c_1$$

We can then see that the pattern is that

$$c_{2k} = \frac{1}{1\cdot 3\cdots(2k-1)}c_0,\ c_{2k+1} = \frac{1}{2\cdot 4\cdots(2k)}c_1$$

Thus the series has solution

$$y = c_0\left(\sum_{k=0}^{\infty}\frac{1}{1\cdot 3\cdots(2k-1)}x^{2k}\right) + c_1\left(\sum_{k=0}^{\infty}\frac{1}{2\cdot 4\cdots(2k)}x^{2k+1}\right)$$

5. From the equation in class, the DE is $my'' + cy' + ky = 0$, or in this case, $2y'' + 6y' + 4y = 0$. This simplifies to $y'' + 3y' + 2y = 0$. This has

auxiliary equation $x^2 + 3x + 2 = 0$, which has roots $x = -1, x = -2$. Thus the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

We know that $y'(0) = 3$, and since the spring is initially in its starting position, we also know $y(0) = 0$. These give equations

$$c_1 + c_2 = 0 \text{ and } -c_1 - 2c_2 = 0$$

which gives $c_1 = 3$, $c_2 = -3$. Thus the position of the spring is given by the equation

$$y = 3e^{-x} - 3e^{-2x}$$

After 1 second, the position will be $y(1) = 3(e^{-1} - e^{-2})$.